For those students of MATH1010I, please be aware of the tutorial format:

Usually, I will dispatch a series of exercises for you to practice. I will give you 45 minutes to do these problems. During the problem session, I will answer all the questions (in putonghua, cantonese or english) that you have regarding these problems or homeworks, lecture notes or any random math problems. The solution to the problems dispatched would be uploaded right after the tutorial. You can hand in your solution to me for marking.

Problem 1. (mathematical induction) Verify that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)\times n}=1-\frac{1}{n}.$$

Solution. We use mathematical induction. We start from from the base case, $n=2$. Obviously, when $n=2$, the left hand side is $\frac{1}{1\times 2}=\frac{1}{2}$. The right hand side is $1-\frac{1}{n}=\frac{1}{2}$ as well. Now assume when $n=k$ that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(k-1)\times k}=1-\frac{1}{k}.$$

When $n=k+1$,

$$\begin{aligned}
&\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(k-1)\times k}+\frac{1}{k\times(k+1)}\\
=&1-\frac{1}{k}+\frac{1}{k(k+1)}\\
=&1-\frac{1}{k}+\frac{(k+1)-k}{k(k+1)}\\
=&1-\frac{1}{k}+\frac{1}{k}-\frac{1}{k+1}\\
=&1-\frac{1}{k+1}.
\end{aligned}$$

So by principle of mathematical induction,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)\times n}=1-\frac{1}{n}.$$

Problem 2. Start from

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

deduce the following:

1. $\sin(A+B)=\sin A\cos B+\cos A\sin B$;
2. $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$.

Solution. 1.

$$\begin{aligned}\sin(A+B)=&\cos(\frac{\pi}{2}-(A+B))\\ =&\cos((\frac{\pi}{2}-A)+(-B))\\ =&\cos(\frac{\pi}{2}-A)\cos(-B)-\sin(\frac{\pi}{2}-A)\sin(-B)\\ =&\sin A\cos B+\cos A\sin B.\end{aligned}$$

2. We have

$$\begin{aligned}\cos 2A=&\cos(A+A)\\ =&\cos A\cos A-\sin A\sin A\\ =&\cos^2 A-\sin^2 A\end{aligned}$$

and

$$\begin{aligned}\sin 2A=&\sin(A+A)\\ =&\sin A\cos A+\cos A\sin A\\ =&2\sin A\cos A.\end{aligned}$$

So

$$\begin{aligned}\tan 2A=&\frac{\sin 2A}{\cos 2A}\\ =&\frac{2\sin A\cos A}{\cos^2 A-\sin^2 A}\\ =&\frac{\frac{2\sin A\cos A}{\cos^2 A}}{\frac{\cos^2 A-\sin^2 A}{\cos^2 A}}\\ =&\frac{2\tan A}{1-\tan^2 A}.\end{aligned}$$

This is the end of solution to Problem 2.

Trignometric functions are of great importance in calculus. We list some elementary formulas in the following table.

$$\cos(A+B)=\cos A\cos B-\sin A\sin B; \tag{1}$$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B; \tag{2}$$

$$\sin 2A=2\sin A\cos A; \tag{3}$$

$$
\begin{aligned}
\cos 2A =& \cos^2 A - \sin^2 A & (4)\\
=& \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1 & (5)\\
=& (1-\sin^2 A) - \sin^2 A = 1 - 2\sin^2 A; & (6)\\
\sin(\pi/2 - A) =& \cos A; & (7)\\
\cos(\pi/2 - A) =& \sin A; & (8)\\
\sin(-A) =& -\sin A; & (9)\\
\cos(-A) =& \cos A; & (10)
\end{aligned}
$$

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x-1| - |x+1|$ for any $x \in \mathbb{R}$.

a) Express the explicit formula of the function f as that of a piecewise defined function, with one 'piece' for $(-\infty, -1), [-1, 1], (1, +\infty)$.

b) Sketch the graph of the function f.

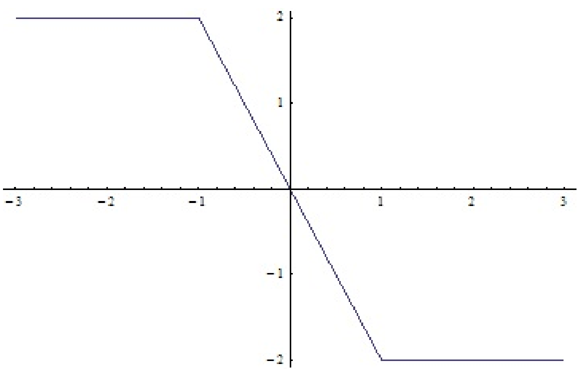
c) Is f an injective function on $\mathbb{R}$? Justify your answer.

d) What is the image of $\mathbb{R}$ under the function f?

Solution. a). There are two absolute values. We break into three intervals, which is $(-\infty, -1)$, $[-1, 1]$ and $(1, +\infty)$. So this will give

$$
f(x) = \begin{cases} (x-1) - (x+1) = -2 & x > 1; \\ -(x-1) - (x+1) = -2x & -1 \leqslant x \leqslant 1; \\ -(x-1) - [-(x+1)] = 2 & x < -1. \end{cases}
$$

b). Using a), we have



c). For a function f, injective mean that $f(x) \neq f(y)$ whenever $x \neq y$. The f given by

$$f(x) = |x-1| - |x+1|$$

is obviously not injective since $f(1) = f(2) = -2$.

d). The image of f is the set $\{f(x): x \in \mathbb{R}\}$. We can see from the graph that the image of f is just

$[-2, 2]$.

Problem 4. Do the same thing as in a) c) and d) for the following functions

e) $f(x) = x^2 - 4\sqrt{x^2} + 3$;

f) $f(x) = ||x-2| - 4|$;

Solution. e). First,

$$\begin{aligned} f(x) =& x^2 - 4\sqrt{x^2} + 3 \\ =& |x|^2 - 4|x| + 3 \\ =& (|x| - 2)^2 - 1. \end{aligned}$$

So

$$f(x) = \begin{cases} (x-2)^2 - 1 & x > 0; \\ (-x-2)^2 - 1 = (x+2)^2 - 1 & x < 0. \end{cases}$$

f is not injective since $f(2) = f(-2) = -1$.

It is obvious from $f(x) = (|x| - 2)^2 - 1 \geqslant -1$. Given any number $a \geqslant -1$, we solve the equation $f(x) = a$, i.e.

$$(|x| - 2)^2 - 1 = a.$$

$x = \sqrt{1+a} + 2$ satisfies $f(x) = a$. So the image of f is $[-1, +\infty)$.

f). First,

$$f(x) = \begin{cases} |x-2| - 4 & |x-2| > 4; \\ 4 - |x-2| & |x-2| \leqslant 4 \end{cases}$$

which is just

$$f(x) = \begin{cases} |x-2| - 4 & x > 6 \text{ or } x < -2; \\ 4 - |x-2| & -2 \leqslant x \leqslant 6. \end{cases}$$

Remove the last remaining absolute value sign,

$$f(x) = \begin{cases} (x-2) - 4 = x - 6 & x > 6; \\ -(x-2) - 4 = -2 - x & x < -2; \\ 4 - (x-2) = 6 - x & 2 \leqslant x \leqslant 6; \\ 4 + (x-2) = 2 + x & -2 \leqslant x < 2. \end{cases}$$

f is not injective since $f(-2) = f(6) = 0$.

f obviously satisfies that $f(x) \geqslant 0$. Given any $a \geqslant 0$, the $f(x) = a$ has a solution(there are more solutions, what are they?)

$$x = 6 + a.$$

So the image of f is $[0, +\infty)$.